Hill's Matrices with Some Vanishing Instability Intervals

Harry Hochstadt
Polytechnic University
Brooklyn, New York 11201

and

Wallace Goldberg
Queens College (CUNY)
Flushing, New York 11367

Submitted by Emeric Deutsch

ABSTRACT

Necessary and sufficient conditions for the vanishing of an instability interval of a Hill's matrix are derived. These lead to algebraic conditions on the entries in the matrix.

The purpose of this article is to provide necessary and sufficient conditions for a Hill's matrix to have multiple eigenvalues. For the general theory we refer to [2]. Previous results regarding the vanishing of instability intervals provide global rather than local conditions (see [3], e.g.). The conditions to be derived here can be translated into algebraic conditions on the entries in the Hill's matrix, and examples wil be given at the end.

A Hill's matrix is an infinite, periodic, symmetric and tridiagonal matrix

$$L=\begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & b_n & a_1 & b_1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & b_1 & a_2 & b_2 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & b_2 & a_3 & b_3 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{1}$$

Here we assume as given $a_1, a_2, \dots a_n, b_1, b_2, \dots, b_n$ such that all a_i, b_i are

52 Vanderbilt Ave., New York, NY 10017 0024-3795/87/$3.50

real, $b_i > 0$, and

$$a_k = a_j, \quad b_k = b_j \qquad \text{if} \quad k \equiv j \bmod n.$$

We now define

$$J_{k,j} = \begin{pmatrix} a_k & b_k & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ b_k & a_{k+1} & b_{k+1} & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{j-1} & b_{j-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_{j-1} & a_j \end{pmatrix} \tag{2}$$

and associate with it two sequences of polynomials: $\{P_k(\lambda)\}$, satisfying

$$P_{k+1}(\lambda) = (\lambda - a_{k+1})P_k(\lambda) - b_k^2 P_{k-1}(\lambda) \tag{3}$$

with initial conditions

$$P_0(\lambda) = 1, \qquad P_1(\lambda) = \lambda - a_1; \tag{4}$$

and $\{Q_k(\lambda)\}$, also satisfying (3) with the initial conditions

$$Q_0(\lambda) = 0, \qquad Q_1(\lambda) = 1. \tag{5}$$

It is easy to verify that

$$|\lambda I - J_{1,n}| = P_n(\lambda), \tag{6}$$

$$|\lambda I - J_{2,n-1}| = Q_{n-1}(\lambda), \tag{7}$$

The zeros of $P_n(\lambda)$ are the eigenvalues of $J_{1,n}$. It is also well known [1] that the above polynomials satisfy

$$P_{k+1}(\lambda)Q_k(\lambda) - P_k(\lambda)Q_{k+1}(\lambda) = -\prod_i^k b_i^2 \tag{8}$$

$$\sum_{j=0}^{k-1} \left[\frac{P_j(\lambda)}{\prod_1^j b_i}\right]^2 = \frac{P_{k-1}(\lambda)\dot{P}_k(\lambda) - \dot{P}_{k-1}(\lambda)P_k(\lambda)}{\prod_1^{k-1} b_i^2} \tag{9}$$

where $\dot{}$ denotes the differentiation with respect to λ.

The discriminant $\Delta(\lambda)$ of the Hill's matrix L is given by

$$\Delta(\lambda)=\frac{P_n(\lambda)-b_n^2Q_{n-1}(\lambda)}{\prod_1^n b_i}. \tag{10}$$

The zeros of $\Delta(\lambda)^2-4$ can be arranged as follows:

$$\lambda_0>\lambda_1\geqslant\lambda_2>\lambda_3\geqslant\lambda_4>\cdots>\lambda_{2n}. \tag{11}$$

The intervals $(\lambda_{2k+1},\lambda_{2k})$ are known as stability intervals, and $(\lambda_{2k+2},\lambda_{2k+1})$ as instability intervals. In the former $|\Delta|<2$, and in the latter $|\Delta|>2$. One can show [2] that in each of the $n-1$ instability intervals there is precisely one eigenvalue of $J_{1,n-1}$, which of course must be a zero of $P_{n-1}(\lambda)$. In the special case when the instability interval vanishes, $\lambda_{2k+2}=\lambda_{2k+1}$, and that point must also be a zero of $P_{n-1}(\lambda)$.

The spectral properties of L must be invariant under cyclic permutations of the subscripts of the a_i and b_i, and the $\{\lambda_k\}$ in (11) denote the spectrum of L. We now consider in analogy to (6), (7)

$$|\lambda I-J_{1+i,n+i}|=P_n^i(\lambda), \tag{12}$$

$$|\lambda I-J_{2+i,n-1+i}|=Q_{n-1}^i(\lambda), \tag{13}$$

and let

$$\Delta(\lambda)=\frac{P_n^i(\lambda)-b_{n+i}^2Q_{n-1}^i(\lambda)}{\prod_1^n b_j}. \tag{14}$$

The $\Delta(\lambda)$ in (10) and (14) must be therefore identical.

We can now state our main result.

THEOREM. *A necessary and sufficient condition on L for the instability interval $(\lambda_{2k+2},\lambda_{2k+1})$ to vanish is that*

$$Q_{n-1}^i(\mu_k)=(-1)^k\frac{\prod_1^n b_j}{b_{n+i}^2},\qquad P_{n-1}^i(\mu_k)=0 \tag{15}$$

for all i.

Proof. To simplify the proof we take $k=0$ and without loss of generality $\mu_k=0$. We also let $\prod_1^n b_j = T$. The proof for general values of k and μ_k is obviously analogous.

We shall first establish the necessity of these conditions. We have

$$\Delta(0) = -2, \qquad \dot{\Delta}(0) = 0. \tag{16}$$

It follows that $P_{n-1}^i(0)=0$ for all i, and from (8), for $k=n-1$, we have

$$P_n^i(0)Q_{n-1}^i(0) = -T^2/b_{n+i}^2 \tag{17}$$

Using (14) and (16) we obtain

$$P_n^i(0) - b_{n+i}^2 Q_{n-1}^i(0) = -2T, \tag{18}$$

and from (18) and (17) we see that

$$Q_{n-1}^i(0) = T/b_{n+i}^2, \tag{19}$$

thus establishing the necessity.

We now turn to the sufficiency of the conditions and assume (15). From (3) we note that

$$P_n^i(0) = -b_{n+i-1}^2 P_{n-2}^i(0),$$

but we observe that $P_{n-2}^i(\lambda) = Q_{n-1}^{i-1}(\lambda)$, so that

$$P_n^i(0) = -b_{n+i-1}^2 Q_{n-1}^{i-1}(0). \tag{20}$$

A combination of (17) and (20) again shows that $P_n^i(0) = -T$ and $Q_{n-1}^{i-1}(0) = T/b_{n+i-1}^2$, so that

$$\Delta(0) = \frac{P_n^i(0) - b_{n+i}^2 Q_{n-1}^i(0)}{T} = -2,$$

showing that 0 is a zero of $\Delta(\lambda)+2$. We now have to demonstrate that it is a double zero. We differentiate (8) at $\lambda=0$, for $k=n-1$, and obtain

$$\dot{P}_n^i(0)Q_{n-1}^i(0) + P_n^i(0)\dot{Q}_{n-1}^i(0) - \dot{P}_{n-1}^i(0)Q_n^i(0) - P_{n-1}^i(0)\dot{Q}_n^i(0) = 0,$$

and it follows that

$$\frac{\dot{P}_n^i(0)T}{b_{n+i}^2} - T\dot{Q}_n^i(0) = 0, \tag{21}$$

since $P_{n-1}^i(0) = 0$ and $Q_n^i(0) = P_{n-1}^i(0) = 0$. Finally we observe that

$$\dot{\Delta}(0) = \frac{\dot{P}_n^i(0) - b_{n+i}^2\dot{Q}_{n-1}^i(0)}{T} = 0 \tag{22}$$

by use of (21). ■

An interesting consequence of the above results is the following.

Corollary. $\dot{\Delta}(\lambda) = [\Sigma_{i=0}^{n-1} Q_n^i(\lambda)]/T$.

Proof. Using the previous notation, we have

$$\dot{\Delta}(\lambda) = \frac{\dot{P}_n^1(\lambda) - b_n^2\dot{P}_{n-2}^2(\lambda)}{T},$$

$$\dot{P}_n^1(\lambda) = \prod_{i=1}^{n} \partial_i P_n^1(\lambda). \tag{23}$$

In the above we view $P_n^i(\lambda)$ as a determinant, and ∂_i represents differentiation of the ith column. Now

$$\partial_i P_n^1(\lambda) = P_{n-1}^{1+i}(\lambda) + b_n^2 P_{n-i-1}^{1+i}(\lambda) P_{i-2}^2(\lambda), \tag{24}$$

where $P_{-1} = 0$. Similarly

$$\dot{P}_{n-2}^2(\lambda) = \sum_{i=1}^{n-2} \partial_i P_{n-2}^2(\lambda) \tag{25}$$

and

$$\partial_i P_{n-2}^2(\lambda) = P_{n-i-2}^{i+2}(\lambda) P_{i-1}^2(\lambda), \tag{26}$$

and by combining the above the result follows. ■

The proof of the theorem involved a single vanishing instability interval. Clearly, if more than one vanishes, several sets of the type displayed in (15) must hold.

We now consider two examples. First we take the case $n = 3$, and as before assume $k = 0$ and $\mu_k = 0$. Then the conditions of the theorem reduce to

$$a_2 b_1 + b_2 b_3 = 0, \qquad a_1 a_2 - b_2^2 = 0,$$

and by a cyclic permutation of the subscripts we find also

$$a_3 b_2 + b_3 b_1 = 0, \qquad a_2 a_3 - b_3^2 = 0,$$

$$a_1 b_3 + b_1 b_2 = 0, \qquad a_3 a_1 - b_1^2 = 0.$$

For $n = 4$ we find

$$\left(a_2 a_3 - b_3^2\right) b_1^2 = b_1 b_2 b_3 b_4, \qquad a_1 a_2 a_3 - a_1 b_3^2 - a_3 b_2^2 = 0.$$

Again one can obtain three more such sets of equations by a cyclic permutation, but we shall refrain from listing them.

REFERENCES

1 N. I. Akhiezer, *The Classical Moment Problem*, Oliver & Boyd, Edinburgh, 1965.
2 H. Hochstadt, On the theory of Hill's matrices and related inverse spectral problems, *Linear Algebra Appl.* 11:41–52 (1975).
3 H. Hochstadt, An inverse spectral problem for a Hill's matrix, *Linear Algebra Appl.* 57:21–30 (1984).

Received 23 February 1987; revised 23 February 1987